Boundary effects to the entanglement entropy and two-site entanglement of the spin-1 valence-bond solid

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(Dated: February 1, 2008)

We investigate the von Neumann entropy of a block of subsystem for the valence-bond solid (VBS) state with general open boundary conditions. We show that the effect of the boundary on the von Neumann entropy decays exponentially fast in the distance between the subsystem considered and the boundary sites. Further, we show that as the size of the subsystem increases, its von Neumann entropy exponentially approaches the summation of the von Neumann entropies of the two ends, the exponent being related to the size. In contrast to critical systems, where boundary effects to the von Neumann entropy decay slowly, the boundary effects in a VBS, a non-critical system, decay very quickly. We also study the entanglement between two spins. Curiously, while the boundary operators decrease the von Neumann entropy of L spins, they increase the entanglement between two spins.

PACS numbers: 75.10.Pq, 03.67.Mn, 03.65.Ud, 05.70.Jk.

I. INTRODUCTION

Recently much research has been undertaken to understand the subtle interplay between *quantum entanglement* and *quantum criticality* for spin systems [1, 2, 3, 4, 5, 6, 7, 8, 9]. Vidal *et al.* [4] showed that the entanglement between a block of contiguous spins and its complement in the ground state of the Ising model shows different behaviours for the gapped and gapless cases (critical and non-critical). The entanglement of the VBS ground state of the much-studied Affleck–Kennedy–Lieb–Tasaki (AKLT) model [10, 11, 12] is considered in Refs. [6, 7], and very recently Campos Venuti *et al.* studied this state's long-distance entanglement property [8]. The entanglement of the fermionic system was studied in Refs.[13, 14].

While much theoretical work in this area has focused on periodic boundary conditions, the open boundary condition has also attracted recent attention [15, 16]. Laflorencie *et al.* [15] numerically studied the boundary effects in the critical scaling of entanglement entropy (von Neumann entropy of a block of spins) for the (gapless) 1D XXZ model, and found the entanglement entropy *slowly* decays away from the boundary with a power-law. This result can be interpreted as stating that in critical systems, the boundary effects to the entanglement entropy is quasi-long-ranged; *i.e.* there is a quasi-long-ranged entanglement between the boundaries and the subsystem in question. This agrees with the fact that entanglement entropy increases logarithmically with the size of the subsystem in critical systems [4, 17]. By contrast, the entanglement entropy for non-critical systems saturates to a constant bound when the subsystem size is increased, implying that the entanglement in the bulk is short-ranged. It is this localised nature of the entanglement entropy around the block edges which gives rise to an area law [17, 18] and makes ground states of gapped 1D systems particuarly amenable to simulation through matrix product states (MPS) [19, 20].

One might expect that boundary effects to the entanglement entropy also have different behaviours for critical and non-critical systems, and an interesting question then is whether the boundary effects to the entanglement entropy are short- or long-ranged, and what the exact behaviour of these are. This is one of the main motivations of this work: here we study the entanglement entropy of a VBS with general open boundary conditions. *We will show that, in contrast with the critical XXZ chain, the boundary effect to the entanglement entropy in the VBS state is short-ranged.* We will also show that the saturated bound for this state is the sum of von Neumann entropy of the two boundaries. Furthermore, it is not a constant (as is the case for a fixed boundary condition); it varies for different boundary conditions. This saturated bound in 1D state corresponds to the area law for higher-dimensional systems.

This model was originally studied by Affleck *et al.* in the context of the Haldane conjecture [10, 11, 12]. It has also been the focus of much renewed interest since its generalisation to MPS—which have been shown to efficiently simulate many 1D systems [19, 20] and may be used as a variational set in density matrix renormalisation group (DMRG) calculations—[21, 22] and the discovery that its analogue in 2D is a resource for universal quantum computation [23]. We hope that studying the boundary effect on the entanglement entropy will give some further insight into this model. Since the boundary effects to the block entanglement entropy decays fast, we expect the the DMRG method can be applied efficiently to this model.

II. DEFINITION OF THE VBS STATE

The spin-1 VBS state with general open boundary conditions (GOBC) [24] takes the form:

$$|\mathrm{VBS}\rangle = Q_l^p \left[\prod_{k=-N_l+1}^{L+N_r-1} (a_k^\dagger b_{k+1}^\dagger - b_k^\dagger a_{k+1}^\dagger)\right] Q_r^q |\mathrm{vac}\rangle \quad (1)$$

where $a_k^\dagger, b_k^\dagger$ are bosonic operators, Q_l^p and Q_r^q are respectively the left and right boundary operators; $p, q = \pm$ with $Q_l^+ = a_{-N_l+1}^\dagger$, $Q_l^- = b_{-N_l+1}^\dagger$, $Q_r^+ = a_{L+N_r}^\dagger$ and $Q_r^- = b_{L+N_r}^\dagger$; $|\mathrm{vac}\rangle$ is the vacuum state; and N_l, N_r are integer numbers. Since the left and right boundary operators are mutually independent, there are altogether four different VBS states with GOBC. Note that all sites in the spin chain including the left and right boundary sites $-N_l+1$, $L+N_r$ are spin-1's. Thus this VBS state (1) is different from that studied in Ref. [7]. We should also note one boundary operator, for example, Q_l^+ changes the boundary state $a_{-N_l+1}^\dagger|vac\rangle$ and $b_{-N_l+1}^\dagger|vac\rangle$. In fact, the state (1) is the ground state of the the Hamiltonian studied by Affleck *et al.* [10],

$$\mathcal{H} = \sum_{j=-N_l+1}^{L+N_r-1} \left[(\mathbf{S}_j \cdot \mathbf{S}_{j+1}) + \frac{1}{3}(\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2\right]. \quad (2)$$

III. DIVIDING THE CHAIN

For convenience in later calculations, we divide this 1D state into three parts: the left-hand, central and right-hand parts. The left-hand part is defined as $|\mathrm{left}, p\rangle = Q_l^p \prod_{k=-N_l+1}^{0} (a_k^\dagger b_{k+1}^\dagger - b_k^\dagger a_{k+1}^\dagger)|\mathrm{vac}\rangle$. Similarly, the right-hand part is defined as $|\mathrm{right}, q\rangle = \prod_{k=L+1}^{L+N_r-1} (a_k^\dagger b_{k+1}^\dagger - b_k^\dagger a_{k+1}^\dagger) Q_r^q |\mathrm{vac}\rangle$. Finally, the central part is written $|\mathrm{central}\rangle = \prod_{k=1}^{L} (a_k^\dagger b_{k+1}^\dagger - b_k^\dagger a_{k+1}^\dagger)|\mathrm{vac}\rangle$. Note that site 1 appears in both the left and central parts, and acts as two spin-1/2's; site L similarly appears in both the central and right parts. Thus the whole VBS state with GOBC now takes the form $|\mathrm{VBS}; p, q\rangle = |\mathrm{left}, p\rangle|\mathrm{central}\rangle|\mathrm{right}, q\rangle$. We should note that this is not strictly a product state, but that this decomposition is valid for our purposes. Double-counting is avoided since each bulk spin consists of two spin-1/2's and the two bosonic operators (spin-1/2's) in one site constitute a spin-1 state by Fock space representation. For example, terms $(a_0^\dagger b_1^\dagger - b_0^\dagger a_1^\dagger)|\mathrm{vac}\rangle$ and $(a_1^\dagger b_2^\dagger - b_1^\dagger a_2^\dagger)|\mathrm{vac}\rangle$ belong to left and central parts, respectively, however, by Fock space representation, the product state will create at site 1 the state $(a_1^\dagger)^2|\mathrm{vac}\rangle, (b_1^\dagger)^2|\mathrm{vac}\rangle$ and $a_1^\dagger b_1^\dagger|\mathrm{vac}\rangle$. Thus the three parts are connected to constitute the original state (1). Our aim is to now study the von Neumann entropy of the reduced density operator of the contiguous spins from site 1 to L of the state $|\mathrm{VBS}; p, q\rangle$. For this aim, according to the theory of entanglement, the left- and right part states $|\mathrm{left}, p\rangle$ and $|\mathrm{right}, q\rangle$ can be replaced by two bipartite states, through the Schmidt decomposition.

Without loss of generality, we start from the left part and consider the entanglement of the quantum state $|\mathrm{left}, p\rangle$ between site 1 and the rest; *i.e.* we consider it a bipartite state with site 1 as one particle and the rest as another particle. According to the Schmidt decomposition, we can first calculate eigenvalues of the reduced density operator of site 1 for state $|\mathrm{left}, p\rangle$.

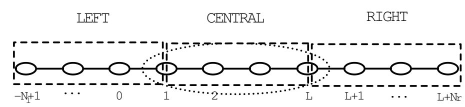

FIG. 1: The quantum spin chain with $L + N_l + N_r$ sites; each is spin-1. We calculate the von Neumann entropy of L spins in the chain. The whole spin chain is divided into three parts: the left, central and right parts. The spin-1 at site 1 is divided into two spin-1/2's; one in the left part, one in the central part. Similarly the spin-1 at site L is split between the central- and right parts.

Denote $|\Psi^-\rangle_{k,k+1} \equiv (a_k^\dagger b_{k+1}^\dagger - b_k^\dagger a_{k+1}^\dagger)|\mathrm{vac}\rangle$, we know $|\Psi^-\rangle_{01}|\Psi^-\rangle_{12} \frac{1}{\sqrt{3}} \sum_{\alpha_k=1}^{3} |\alpha_1\rangle (I \otimes \sigma_{\alpha_1})|\Psi^-\rangle_{0,2}$, where $\{\sigma_i\}_{i=0}^3$ is the Pauli group, and we have defined the states $|\alpha_1\rangle = I \otimes \sigma_{\alpha_1}|\Psi^-\rangle$. Here $\sigma_1 = a_1^\dagger b_1 + a_1 b_1^\dagger$, $\sigma_2 = -\mathrm{i} a_1^\dagger b_1 + \mathrm{i} a_1 b_1^\dagger$, $\sigma_3 = a_1^\dagger a_1 - b_1^\dagger b_1$ and $\sigma_0 = a_1^\dagger a_1 + b_1^\dagger b_1$, the identity. By this result, the state of the left part may be written

$$|\mathrm{left}, p\rangle = \frac{1}{3^{(N_l-1)/2}} \sum_{\alpha_0,\ldots,\alpha_{-N_l+2}=1}^{3} |\alpha_{-N_l+2}\rangle \otimes \cdots \otimes |\alpha_0\rangle$$
$$\times\, (Q_l^p \otimes \sigma_{\alpha_0} \cdots \sigma_{\alpha_{-N_l+2}})|\Psi^-\rangle_{-N_l+1,1}. \quad (3)$$

It is now possible to calculate the site 1 reduced density operator. Using the identity $\sum_{\alpha=1}^{3} (I \otimes \sigma_\alpha)|\Psi^-\rangle\langle\Psi^-|(I \otimes \sigma_\alpha)^\dagger = I - |\Psi^-\rangle\langle\Psi^-|$ (where I on the l.h.s. and r.h.s. is the identity in $\mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^2$, respectively) we find

$$\rho_1 = \mathrm{Tr}_1 (Q_l^p \otimes I)[\frac{1}{4}(1 - f_l)I + f_l |\Psi^-\rangle\langle\Psi^-|](Q_l^p \otimes I)^\dagger, (4)$$

where $f_l = (-\frac{1}{3})^{N_l-1}$, and the trace is over the first Hilbert space. Now we find that the matrix form of the reduced density operator of site 1 takes a diagonal form $\rho_1 = \mathrm{diag}(\xi_l^+, \xi_l^-)$, where we have defined $\xi_l^\pm = (3 \pm f_l)/3$ and $f_l = (-1/3)^{N_l-1}$. For different boundary operators $Q_l^\pm$, state ρ_1 is invariant under a basis transformation. By entanglement theory, we can replace the quantum state of left part by a real bipartite state $|\mathrm{left}, p\rangle \rightarrow |\phi_l\rangle \equiv \left(\sqrt{\xi_l^+} a_0^\dagger b_1^\dagger - \sqrt{\xi_l^-} b_0^\dagger a_1^\dagger\right)|\mathrm{vac}\rangle$.

We find that the reduced density operator ρ_1 converges to the identity exponentially fast with respect to N_l, and thus we can simply consider $|\phi_l\rangle$ as a singlet state $|\Psi^-_{0,1}\rangle$ when $N_l \to \infty$. Similarly for right part of the state, we have $|\phi_r\rangle \equiv \left(\sqrt{\xi_r^+} a_L^\dagger b_{L+1}^\dagger - \sqrt{\xi_r^-} b_L^\dagger a_{L+1}^\dagger\right)|\text{vac}\rangle$, where the $\xi_r^\pm$ have a similar definition to the $\xi_l^\pm$. Thus the VBS state in Eq. (1) may be rewritten

$$|\text{VBS}\rangle = |\phi_l\rangle \prod_{k=1}^{L-1}(a_k^\dagger b_{k+1}^\dagger - b_k^\dagger a_{k+1}^\dagger)|\phi_r\rangle|\text{vac}\rangle, \quad (5)$$

where indices p, q are suppressed since they do not change the result. The validity of the transformation from (1) to (5) in studying the von Neumann entropy of contiguous L spins can also be checked by a method with matrix product state representation introduced in, for example, Ref.[22]. A numerical calculation for small L confirms our result.

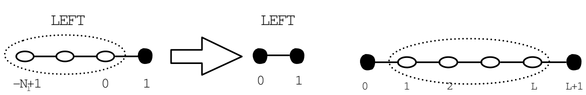

FIG. 2: The quantum state of left part is considered as a bipartite state with one particle in site 1 and the rest as another particle. According to the Schmidt decomposition this state is equal to a bipartite state $|\phi_l\rangle$, and each spin is spin-1/2 at site 0 and 1; Finally, the VBS state with GOBC is mapped to a state with $L+2$ sites with two spin-1/2 boundaries at ends 0 and $L+1$

In the case that $N_l \to \infty$, $N_r \to \infty$ the state takes the form $|\text{VBS}\rangle = \prod_{k=0}^{L}(a_k^\dagger b_{k+1}^\dagger - b_k^\dagger a_{k+1}^\dagger)|\text{vac}\rangle$. The entanglement entropy of this state has been studied previously [7]; it was found that there are no boundary effects. In this paper, one of our main concerns is to show that the VBS state (1) *does* have a boundary effect. Already we know that if the block of L contiguous spins in Eq. (1) is far from the two boundary sites the boundary effect will decay very rapidly (exponentially). We now present explicitly the entanglement entropy of these L spins. Let us first rewrite the left part state in the form $|\phi_l\rangle = (V_l \otimes I)|\Psi^-\rangle_{01}$, where the matrix form of V_l takes the form $V_l = \text{diag}(\xi_l^+, \xi_l^-)$. Note that V_l is not necessarily unitary. In a similar manner for right part state , V_r can also be defined $V_r = \text{diag}(\xi_r^-, \xi_r^+)$, giving $|\phi_r\rangle = (I \otimes V_r)|\Psi^-\rangle_{L,L+1}$. We can then write the VBS with GOBC in the form:

$$\begin{aligned}|\text{VBS}\rangle \;&=\; (V_l \otimes V_r)_{0,L+1} \sum_{\{\alpha_i\}} |\alpha_1\rangle \otimes \cdots \otimes |\alpha_L\rangle \\ &\quad \times (\sigma_{\alpha_1} \cdots \sigma_{\alpha_L} \otimes I)|\Psi^-\rangle_{0,L+1}, \quad (6)\end{aligned}$$

where the summation is from 1 to 3 for indices $\alpha_1, \cdots, \alpha_L$. According to entanglement theory, the von Neumann entropy of the reduced density operator of L spins is the same as the von Neumann entropy of the two ends (sites 0 and $L+1$). The reduced density operator of these sites takes the form

$$\tilde{\rho}_L = (V_l \otimes V_r)\left\{\frac{1-p}{4} I + p|\Psi^-\rangle\langle\Psi^-|\right\}(V_l \otimes V_r)^\dagger \quad (7)$$

where $p = (-1/3)^L$.

Expanding this, we find the following form for the density matrix:

$$\tilde{\rho}_L = \begin{pmatrix} \frac{1-p}{4}(\xi_l^+\xi_r^-)^2 & 0 & 0 & 0 \\ 0 & \frac{1+p}{4}(\xi_l^+\xi_r^+)^2 & -\frac{p}{2}\xi_l^+\xi_l^-\xi_r^+\xi_r^- & 0 \\ 0 & -\frac{p}{2}\xi_l^+\xi_l^-\xi_r^+\xi_r^- & \frac{1+p}{4}(\xi_l^-\xi_r^-)^2 & 0 \\ 0 & 0 & 0 & \frac{1-p}{4}(\xi_l^-\xi_r^+)^2 \end{pmatrix}$$

When p is small the eigenvalues of the matrix $\tilde{\rho}_L$ can be found by a Taylor expansion to be $\lambda_1 = \xi_l^+\xi_r^-(1-p)$; $\lambda_2 = \xi_l^-\xi_r^+(1-p)$; $\lambda_3 = \xi_l^+\xi_r^+(1+p) + O(p^2)$; $\lambda_4 = \xi_l^-\xi_r^-(1+p) + O(p^2)$. Recall that $|\phi_l\rangle$ is a pure state, so the reduced density operators at sites 0 and 1 are the same under unitary transformation (*i.e.* $\rho_0 = \rho_1$); similarly for state $|\phi_r\rangle$. By checking the eigenvalues of $\tilde{\rho}_L$, we find that $\tilde{\rho}_L = \rho_0 \otimes \rho_{L+1} + O(p)$, where the equation is true under a unitary transformation. This transformation has no effect on the von Neumann entropy, and thus we suppress it. The density operator of $\tilde{\rho}_L$ converges exponentially fast to the tensor product of two ends; the speed is $p = (-1/3)^L$.

IV. BLOCK ENTROPY

Finally we calculate explicitly the von Neumann entropy of a block of L spins,

$$S(\tilde{\rho}_L) = S(\rho_0) + S(\rho_{L+1}) + O(p). \quad (8)$$

The von Neumann entropy of L spins converges to the von Neumann entropy of two ends exponentially fast with convergence $p = (-1/3)^L$. The exact form of $O(p)$ is written below [31]. For a pure bipartite state $|\phi_l\rangle$, we

know $S(\rho_0) = S(\rho_1) = -\frac{3-f_l}{6}\log\frac{3-f_l}{6} - \frac{3+f_l}{6}\log\frac{3+f_l}{6}$. When f_l is small, we find $S(\rho_0) \sim 1 - \frac{f_l^2}{18}$ and similarly for $S(\rho_{L+1})$. So when the distances between the block of L contiguous spins and the two ends are large, we know that

$$S(\tilde{\rho_L}) \sim 2 - \frac{f_l^2 + f_r^2}{18} + O(p) + O(f_l^4) + O(f_r^4). \quad (9)$$

Note that when p is comparable with f_l and f_r, then $O(p)$ may have contributions for both f_l and f_r. Considering all of these points, we conclude that boundary effects decay exponentially when the distances between the subsystem and the boundaries increase. In the case that there are no boundary operators, the trace in Eq. (4) is over the identity, and ρ_1 reduces to I. Hence the boundary effects to the entanglement entropy never arise; this is the case previously studied [7]. We remark for 1D VBS with GOBC, the terms corresponding to the topological entropy in Ref.[25, 26] are the boundary terms appeared in Eq.(9), and they remain unchanged for $L \to \infty$.

V. COMPARISON WITH A CRITICAL SYSTEM

It was found numerically [15] that for spin-1/2 XXZ chains the entropy takes the form $S(L,N) = S_U(L,N) + (-1)^L S_A(L,N)$, where the second term arises from the boundary conditions, and can be written $S_A(L,N) = 1/(\sin(2\pi N_{n,r}/N)N/\pi)^K$. Note that the notation of Ref. [15] has been changed slightly here and $N = N_r + N_l + L$, $N_l = N_r = N_{n,r}$; K depends on the anisotropy parameter in the XXZ chains and $K = 1$ for an XX chain. We find that the boundary term decays slowly and is quasi-long-ranged, while for the VBS studied in this paper, the boundary terms in Eq. (9) decay exponentially fast.

VI. TWO-SITE ENTANGLEMENT BY NEGATIVITY AND REALIGNMENT CALCULATIONS

We next study the boundary effects to entanglement between only *two spins* in the bulk. Consider spins 1, L. The previous method still works; *i.e.* we can reduce the length of the chain from $N_l + L + N_r$ sites in Eq. (1) to a chain with only $L+2$ spins in Eq. (5). This transformation does not change the entanglement between two spins at sites 1 and L. We find that the density operator of the bipartite state is written

$$\rho_{1,L} = \sum |\alpha_1\alpha_L\rangle\langle\alpha_1'\alpha_L'| \operatorname{Tr}(V_l\otimes V_r)(\sigma_{\alpha_1}\otimes\sigma_{\alpha_L}^t) \times \left[\frac{1-p}{4}I + p|\Psi^-\rangle\langle\Psi^-|\right](\sigma_{\alpha_1'}\otimes\sigma_{\alpha_L'}^t)^\dagger(V_l\otimes V_r)^\dagger \quad (10)$$

The explicit form of this density matrix is complicated. However since the boundary matrices V_l and V_r become identities exponentially with N_l and N_r, we know that the boundary effects to this density operator decay exponentially if the separation of these two spins with the boundaries increase. Thus the case that N_r or N_l is small already can provide enough information about the entanglement between the considered two spins. The cases $N_r \to \infty$ and $N_l \to \infty$ are special since they correspond to the cases that there are no boundary operators $Q_r^\pm$ and $Q_l^\pm$, respectively.

	$N_r = 1$	$N_r = 2$	$N_r = 3$	$N_r = 4$	$N_r = \infty$
$N_l = 1$	1.45919	1.50111	1.43456	1.45142	1.44170
$N_l = 2$	1.50111	1.05433	1.15504	1.11552	1.12486
$N_l = 3$	1.43456	1.15504	1.00609	1.05018	1.03861
$N_l = 4$	1.45142	1.11552	1.05018	1.00068	1.01252
$N_l = \infty$	1.44670	1.12486	1.03861	1.01252	1 (exact)

TABLE I: Nearest-neighbour negativity

	$N_r = 1$	$N_r = 2$	$N_r = 3$	$N_r = 4$	$N_r = \infty$
$N_l = 1$	0.37393	0.23445	0.19692	0.20032	0.19861
$N_l = 2$	0.23445	0.03974	0.02631	0.02194	0.02221
$N_l = 3$	0.19692	0.02631	0.00439	0.00293	0.00247
$N_l = 4$	0.20032	0.02194	0.00293	0.00049	0.00027
$N_l = \infty$	0.19861	0.02221	0.00247	0.00027	0 (exact)

TABLE II: Nearest-neighbour entanglement by realigment

In order to quantify the two-spin entanglement in this case, we shall use *negativity*, $\mathcal{N}$ [27]. We find that there is no entanglement for non-nearest neighbouring spins (regardless of boundary). The nearest-neighbour negativity for a range of N_l, N_r is presented in Table I, where a factor 1/9 is omitted.

Curiously, we see that while boundary operators *decrease* the block von Neumann entropy—see Eq.(8)—they *increase* the nearest-neighbour negativity. Roughly, this can be understood that the entanglement is monogamous [29, 30], i.e., it can not be shared freely by many parties. A simple example about the monogamy of entanglement is that, suppose A, B and C are three parties, if A and B are maximally entangled, A and C will be separable. The result in this paper shows that the boundary operators have effect on the entanglement sharing in this many-body system.

In the table above, we see that for finite $N_{l,r}$, it is always that case that $\mathcal{N} > 1/9$. Asymmetric boundary operators may also increase the entanglement. For example, $\mathcal{N} = 1.45919/9$ for $N_l = N_l = 1$, but $\mathcal{N} = 1.50111/9$ for $N_l = 1, N_r = 2$. Now $\mathcal{N}$ does not vary monotonically with $N_{l,r}$; *e.g.* when N_r varies from 1 to ∞ (for $N_l = 1$), $\mathcal{N}$ oscillates. This seemingly new result may potentially be useful: nearest-neighbour entanglement may be controlled by tweaking boundaries.

The drawback of the *negativity* is that the bound entanglement cannot be detected and quantified. A complementary quantity derived from the realignment separability criterion can partially solve this problem [28].

For a two-site density operator ρ, this quantity is defined as $\mathcal{R} = (||R(\rho)||-1)/2$, where the matrix $R(\rho)$ is obtained from the density operator ρ by the realignment method, and $||\cdot||$ is the trace norm. The larger of the quantities $\mathcal{N}$ and $\mathcal{R}$ gives a lower bound of the *concurrence* $\mathcal{C}$ for a mixed state in arbitrary dimensional systems; *i.e.* $\mathcal{C} \geq \max\{\mathcal{N}, \mathcal{R}\}$ [28].

Using this measure, we still find zero entanglement between detected for non-nearest neighbouring spins regardless of boundary. The entanglement of nearest neighbouring spins by realignment is presented in Table II and a factor 1/9 is also omitted. We also still find that the boundary operators increase entanglement $\mathcal{R}$, and indeed no entanglement is found for the case without boundary operators: this is due to the limitation of the realignment method. Finally let us remark that for the model studied in this paper, the *negativity* provides a stronger lower bound for the *concurrence*. Since the concurrence for a general mixed state in $\mathcal{C}^3 \otimes \mathcal{C}^3$ is difficult to find, the entanglement measures by *negativity* and *realignment* are widely accepted.

VII. CONCLUSION

In summary, we studied the boundary effects to the entanglement entropy and the two-site entanglement for the spin-1 VBS state which is the ground state for a gapped model. We showed that the boundary effects are short-ranged, *i.e.* they decays exponentially in the distance between the subsystem considered and the boundary sites. This is different from the case of XXZ chain which is a gapless model. The two-site entanglement was studied by two entanglement measures, the negativity and the realignment method. For the VBS state, we find the boundary operators decrease the block von Neumman entropy but increase the the nearest-neighbour entanglement measured by negativity and realignment.

VIII. ACKNOWLEDGMENTS

H.F. was supported by 'Bairen' program, NFSC and '973' program (2006CB921107), and V.K. was supported in part by NSF Grant DMS 0503712. C.H. and S.B. acknowledge UK EPSRC grants EP/P500559/1 and GR/S62796/01. We thank F. Verstraete for useful discussions.

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